signed in complying with a physical beam model. In the simulations presented here, we assumed that all states and their first time derivatives were available through an appropriate measurement system. This is the same assumption made in Ref. 7.

The simulation result depicted in Fig. 3 shows the estimated natural frequency of the system. It is worth noting that the initial frequencies were intentionally given an unreasonable magnitude order $[\hat{\omega}_0, \hat{\omega}_1]^T = [14.0, 3.0]^T$ rad/s such that the flexural frequency is much lower than that of the rigid mode. For this particular example, the neural estimator demonstrates the capability to learn the frequency spectrum of the system accurately within 3000 iterations. Moreover, the learning time could be reduced if more accurate initial guess values were provided. The learning rate may vary significantly from one application to another depending on the complexity of the problem. The simulation result in Fig. 4 shows the estimation of system inertia $\hat{I}_H$ and flexural modal gain $d\hat{Y}_1(0)/dx$ where unitary initial values are assigned to both parameters without loss of generality. It is observed that both parameters converge to true values much faster than the flexural frequency estimation. The estimated modal damping $\hat{\xi}_0, \hat{\xi}_1$ also converged to the correct values within about the same number of iterations.

IV. Conclusions

In this paper, a neural-network-based estimator is used for the modal parameter estimation of a flexible beam. The neural estimator is designed based on a modified form of the Hopfield network architecture. Simulation results demonstrate the asymptotic convergence in the state of neurons to the modal parameters of a case study without prior knowledge, assuming that the system states and its time derivatives are available. The proposed estimator has the following advantages over the traditional parameter estimators such as recursive least squares: the network model, and therefore the estimator itself, is a nonlinear dynamic system. It has the potential to handle the parameter estimation problem of nonlinear systems. With the analog circuit implementation of Hopfield nets, it can be implemented in real time and be applied to highly nonlinear systems.

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Evaluation of Inertial Integrals for Multibody Dynamics

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Introduction

When a continuum representation is employed in the development of multibody dynamics equations (e.g., Ref. 1), a class of inertial integrals must be evaluated to enable numerical simulation of the dynamical system. For all but the simplest cases, direct and explicit evaluation of these integrals is impractical; some form of approximation is generally employed. We herein present an approach for the evaluation of the needed integrals in terms of readily available finite element derived quantities. Because of our particular choice of modes, the integral evaluation requires no further approximations than those inherent to the finite element analysis and the model reduction. The inertial representation of the individual bodies may involve either lumped or consistent mass matrices.

The approach developed involves evaluating the kinetic energy of a vibrating body via two distinct methods, followed by a transformation to a common set of independent generalized velocities. Expressions for the required inertial integrals follow from equating the matrices of the respective quadratic forms.

Displacement Field Representation

System Kinematics

Consider a generic flexible body whose kinematics are defined with the aid of two frames: an inertial frame $\boldsymbol{F}_n$ and an embedded-body frame $\boldsymbol{F}_b$. In the present context, all flexible and rigid-body translations/rotations are assumed to be small. This assumption is an expedient to the integral evaluation. As will be seen, the end result is applicable to both large and small displacements of a flexible body. The absolute displacement (not position) of a generic point of the flexible body is expressed in the form

$$d(r,t) = [I^3 \tilde{r}^T]\, U_o(t) + u(r,t), \quad U_o(t) = \begin{Bmatrix} u_o(t) \\ \theta_o(t) \end{Bmatrix} \tag{1}$$

where

$$\tilde{r} \equiv \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}; \quad r \equiv \begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix} \tag{2}$$

Here and in the sequel, for any integer p, I^p denotes the p-dimentional identity matrix.

The vector u_o denotes the translation of the origin of $\boldsymbol{F}_b$ relative to that of $\boldsymbol{F}_n$, r denotes the location of a generic point before deformation relative to the origin of $\boldsymbol{F}_b$, and $u(r, t)$ represents the deformation field. Since the body frame is embedded, the boundary conditions are $u(0, t) = 0$ and curl $u(0, t) = 0$. The triplet θ_O denotes the successive small rotations of the three axes of $\boldsymbol{F}_b$ relative to those of $\boldsymbol{F}_n$. All quantities are expressed in the body frame.

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Discrete Domain Displacement Field (Finite Elements)

From finite element theory, the continuous displacement field representation of a flexible body may be written in the form

$$u(r,t)^{3\times 1} = \vartheta(r)^{3\times n}\, q(t)^{n\times 1} \tag{3}$$

where $\vartheta(r)$ is a matrix of interpolation functions, and $q(t)$ is a column matrix which provides translation and rotation data at a discrete set of points (nodes), relative to $\boldsymbol{F}_b$. The node located at the origin of $\boldsymbol{F}_b$ is excluded from $q(t)$ since all relative generalized displacements vanish at this point. Specifically, $q(t)$ has the form

$$q(t)^{n\times 1} = \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \end{Bmatrix}; \quad U_i = \begin{Bmatrix} u_i \\ \theta_i \end{Bmatrix} \equiv \begin{Bmatrix} \text{nodal translations} \\ \text{nodal rotations} \end{Bmatrix} \tag{4}$$

As expressed in Eq. (3), the displacement field is in general written in terms of a high number of generalized coordinates. Model reduction is invoked as

$$q^{n\times 1}(t) = \Phi^{n\times m}\eta^{m\times 1}(t), \quad \{m \le n\} \tag{5}$$

We make the important observation that the matrix Φ is largely arbitrary.

Continuous Domain

When the dynamics equation development follows a continuum approach, the elastic displacement is expressed in the form

$$u^{3\times 1}(r,t) = \Psi^{3\times m}(r)\,\eta^{m\times 1}(t) \tag{6}$$

where $\Psi(r)$ and $\eta(t)$, respectively, denote a matrix of admissible modes and a modal coordinate vector. We note that the modal matrix $\Psi(r)$, like Φ, is arbitrary to a large degree. It is up to the analyst to concoct a set of admissible functions.

Choice of Modes

We now proceed to remove the guesswork involved in choosing $\Psi(r)$ by inspecting Eqs. (3), (5), and (6), and observing that it is perfectly acceptable to let

$$\Psi^{3\times m}(r) \equiv \vartheta(r)^{3\times n}\Phi^{n\times m}; \quad \eta^{m\times 1}(t) \equiv \eta^{m\times 1}(t) \tag{7}$$

As a beneficial consequence of Eq. (7), we observe that the displacement field which is produced by the continuum method [Eq. (6)] and by the finite element method [Eqs. (3) and (5)] are not approximations of each other, but rather are exactly equal—leading to no approximation beyond those inherent to finite element analysis and model reduction.

Kinetic Energy Development

Finite Elements/Discrete Domain

Let q_a denote the absolute (relative to $\boldsymbol{F}_n$) generalized displacements at the finite element nodes. The six nodal degrees of freedom associated with the origin of $\boldsymbol{F}_b$ are included in q_a, and in our convention are assigned to its first six elements.

Employing Eqs. (1), (3), and (5), it can be shown that

$$q_a^{(n+6)\times 1}(t) = \begin{bmatrix} I^6 & 0 \\ R & \Phi \end{bmatrix} \begin{Bmatrix} U_o^{6\times 1} \\ \eta^{m\times 1} \end{Bmatrix}; \quad R^{n\times 6} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \end{bmatrix}; \quad R_i = \begin{bmatrix} I^3 & \tilde{r}_i^T \\ 0 & I^3 \end{bmatrix} \tag{8}$$

With the availability of the free-free mass matrix $M^{(n+6)\times(n+6)}$ the system kinetic energy can be written as

$$T(t) = (1/2)\,\dot{q}_a^T M\, \dot{q}_a, \quad M = \begin{bmatrix} M_{oo}^{6\times 6} & M_{oi} \\ M_{io} & M_{ii}^{n\times n} \end{bmatrix} \tag{9}$$

Combining Eqs. (8) and (9), we write

$$T(t) = \frac{1}{2} \begin{Bmatrix} \dot{U}_o^{6\times 1} \\ \dot{\eta}^{m\times 1} \end{Bmatrix}^T \overline{M} \begin{Bmatrix} \dot{U}_o^{6\times 1} \\ \dot{\eta}^{m\times 1} \end{Bmatrix}; \quad \overline{M} = \begin{bmatrix} I_o & M_{of} \\ (\text{sym}) & M_{ff} \end{bmatrix} \tag{10}$$

with

$$I_o = M_{oo} + M_{oi}R + R^T M_{io} + R^T M_{ii} R \tag{11}$$

$$M_{of} = M_{oi}\Phi + R^T M_{ii}\Phi \tag{12}$$

$$M_{ff} = \Phi^T M_{ii}\Phi \tag{13}$$

where the quantities are of familiar form and can be evaluated easily using general purpose structural analysis codes.

Continuous Domain

The kinetic energy is now expressed as

$$T(t) = \frac{1}{2}\int_{\text{vol}} \rho(r)\, \dot{d}^T(r,t)\, \dot{d}(r,t)\, \mathrm{d}v \tag{14}$$

where $\rho(r)$ is the volumetric mass density. Combining Eqs. (1), (6), (7), and (14), the kinetic energy takes the explicit form

$$T(t) = \frac{1}{2} \begin{Bmatrix} \dot{u}_o \\ \dot{\theta}_o \\ \dot{\eta} \end{Bmatrix}^T \int_{\text{vol}} \rho(r) \begin{bmatrix} I^3 & \tilde{r}^T & \Psi \\ & \tilde{r}\tilde{r}^T & \tilde{r}\Psi \\ \text{sym} & & \Psi^T\Psi \end{bmatrix} \mathrm{d}v \begin{Bmatrix} \dot{u}_o \\ \dot{\theta}_o \\ \dot{\eta} \end{Bmatrix} \tag{15}$$

Integral Identification

In a direct way, it is now possible to identify the integrals which typically appear in the continuum formulation. We compare Eqs. (10) and (15), which leads to

$$\int_{\text{vol}} \rho(r) \begin{bmatrix} I^3 & \tilde{r}^T \\ \tilde{r} & \tilde{r}\tilde{r}^T \end{bmatrix} \mathrm{d}v = I_o; \qquad \int_{\text{vol}} \rho(r) \begin{bmatrix} \Psi \\ \tilde{r}\Psi \end{bmatrix} \mathrm{d}v = M_{of} \tag{16}$$

and

$$\int_{\text{vol}} \rho(r)\, \Psi^T\Psi\, \mathrm{d}v = M_{ff} \tag{17}$$

where, we recall, the right-hand-side quantities are easily obtained from general purpose structural analysis codes. Equations (16) and (17), along with Eqs. (11–13), express the required inertial integrals in terms of readily available finite element data. We observe at this point that the inertial integrals [Eqs. (16) and (17)] are independent of the body state of motion. Thus, our evaluation remains valid in the context of large rigid-body translations and rotations.

Table 1 Inertial integrals [convergence relative to number of elements (n)]

n	M_{0f}					M_{ff} (diagonal elements)				
10	0.6353,	−0.2083,	0.1207,	−0.0816,	0.0585,	0.4979,	0.4818,	0.4512,	0.4090,	0.3594,
100	0.6366,	−0.2122,	0.1273,	−0.0909,	0.0706,	0.5000,	0.4998,	0.4995,	0.4990,	0.4983,
Exact	0.6366,	−0.2122,	0.1273,	−0.0909,	0.0707,	0.5000,	0.5000,	0.5000,	0.5000,	0.5000,
	$\left(\frac{2}{\pi},\right.$	$\frac{-2}{3\pi},$	$\frac{2}{5\pi},$	$\frac{-2}{7\pi},$	$\left.\frac{2}{9\pi}\right)$	$\left(\frac{1}{2},\right.$	$\frac{1}{2},$	$\frac{1}{2},$	$\frac{1}{2},$	$\left.\frac{1}{2}\right)$

Reference 2 presents an algorithm for the evaluation of the preceding integrals (and others). In contrast to our treatment, no apparent physical argument is provided, and the resulting algorithm is exceedingly more complicated.

General Observations

As indicated, there is a great deal of latitude in the choice of Φ. The only requirement set forth herein is that it conform to the body frame $\boldsymbol{F}_b$ embedded in the elastic body. The abundant literature in the area of component mode synthesis offers a diverse set of candidates for Φ [e.g., cantilevered normal modes (Ref. 3) or loaded-interface modes (Ref. 4)]. Further discussion of this issue is beyond the scope of this treatment. We also note that, depending on the degree of linearization, other inertial integrals, not addressed in this Note, will appear in the motion equations.

Example

We will illustrate the application of the inertial integral evaluations [Eqs. (16) and (17)] for the simple case of an axially vibrating rod. Since this problem is one dimensional, the matrices I_0 and M_{0f} degenerate into a scalar and row vector, respectively. The computations are simple and can be readily compared with the direct evaluation of the inertial integrals by employing the exact eigenfunctions of the vibration problem.

The rod is assumed to be long and slender with volumetric mass density ρ and axial stiffness EA. These material and geometric properties are uniform over the length (unstrained) L of the rod. Here, $u(x,t)$ denotes the one-dimensional elastic displacement field with $0 \le x \le L$. The rod is divided into n finite elements, each of length $l = L/n$. Employing a linear interpolation between the nodes, and assembling the element stiffness and mass matrices, we arrive at the unconstrained system stiffness matrix $K^{n+1 \times n+1}$ and mass matrix $M^{n+1 \times n+1}$.

$$[K]_{ij} = \begin{cases} EA/l, & i = j = 1 \text{ or } i = j = n+1 \\ 2EA/l, & i = j \text{ and } 2 \le i \le n \\ -EA/l, & |i-j| = 1 \\ 0, & |i-j| > 1 \end{cases}$$

$$[M]_{ij} = \begin{cases} \rho Al/3, & i = j = 1 \text{ or } i = j = n+1 \\ 2\rho Al/3, & i = j \text{ and } 2 \le i \le n \\ \rho Al/6, & |i-j| = 1 \\ 0, & |i-j| > 1 \end{cases} \tag{18}$$

Corresponding to a unit displacement at $x = 0$ (node 1), the rigid-body mode has unit displacements at the remaining nodes (2 through $n+1$)—hence $R^{n \times 1} = (1, 1, \ldots, 1)^T$. Partitioning the matrix in Eq. (18) as in Eq. (9) (and recalling the degeneracy), we have $M_{00} = \rho AL/(3n)$ and $M_{0i}^{1 \times n} = \rho AL\,[(1/6n), 0, 0, \ldots, 0]$. The matrix $M_{ii}^{n \times n}$ is obtained by deleting the first row and column from M. We take for $\Phi^{n \times m}$ the first m fixed-free modes of the vibrating bar. These modes are the eigenvectors $\boldsymbol{v}$ of the generalized eigenvalue problem $K_{ii}\boldsymbol{v} = \omega^2 M_{ii}\boldsymbol{v}$ where $K_{ii}^{n \times n}$ is the matrix obtained by deleting the first row and column from $K^{n+1 \times n+1}$. For convenience, we express the preceding stiffness and mass matrices as $K_{ii} = (EA/L)K'$ and $M_{ii} = \rho ALM'$, and note that the matrices K' and M' can be computed once the number of elements n is specified. Solving the eigenvalue problem $K'\boldsymbol{v} = \lambda M'\boldsymbol{v}$, we form the matrix $\Phi_b \equiv [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m]$ whose columns are the eigenvectors corresponding to the first (smallest) m eigenvalues. The estimated natural frequencies are given by $\omega_i = \sqrt{(E/\rho L^2)}\,\lambda_{(i=1,2,\ldots,n)}^{1/2}$. Performing the calculations indicated in Eq. (11) yields $I_0 = \rho AL$—the rod mass, in agreement with the left-hand side of Eq. (16).

Numerical results were generated using 10 and then 100 elements. Increasing the value of m (the number of retained modes) increases the order of the matrices $M_{0f}^{1 \times m}$ and $M_{ff}^{m \times m}$, but does not alter the previously obtained values. We also note that for the modes generated by the eigenvalue problem, the matrix M_{ff} [see Eq. (13)] will be diagonal. The eigenvectors in Φ were normalized such that the amplitudes at the end of the rod ($x = L$) are unity. Table 1 gives numerical results for the two inertial integrals M_{0f} and M_{ff} as calculated by Eqs. (12) and (13), respectively. It is of interest to compare these results with those obtained directly from the left-hand sides of Eqs. (16) and (17) when employing the exact eigenfunctions for the vibrating rod in Ψ_c. These eigenfunctions are given by $\psi_i(x) = (-1)^{i+1}\sin(2i-1)\pi x/2L$ (Ref. 5) where the normalization has been done in the same sense as in the columns of Φ. Performing the integrations, we find $M_{0f} = \rho AL\ (2/\pi, -2/3\pi, 2/5\pi, \ldots)$ and $M_{ff} = \rho AL$ diag $(1/2, 1/2, 1/2, \ldots)$. For purposes of comparison, these results are indicated numerically in the last row of Table 1. The convergence of the finite element solutions to these values is apparent.

Conclusion

A simple technique is presented to evaluate the inertial integrals which arise in the flexible dynamics modeling of nonlinear multibody systems. The derivation provides the link between the continuum representation of a flexible body and the practical implementation of the finite element method. It is shown that the inertial integral evaluation introduces no approximations beyond those inherent to the finite element analysis if one adopts a particular basis in the continuum expansion of the elastic displacement field.

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